Generalized Cut-Set Bounds for Broadcast Networks

Tie Liu

Joint work with Amir Salimi and Shuguang (Robert) Cui

Explicit Network Coding Bounds

- Functional-dependency relationship
- Entropy inequalities

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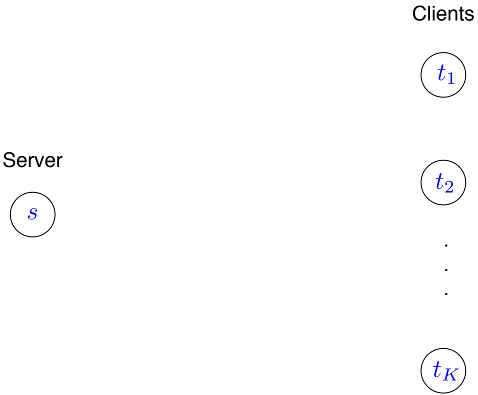
- Functional-dependency relationship
  - Cut structure of the network
- Entropy inequalities
  - Shannon-type inequalities

Broadcast Networks

- Motivation: Content delivery over wired networks



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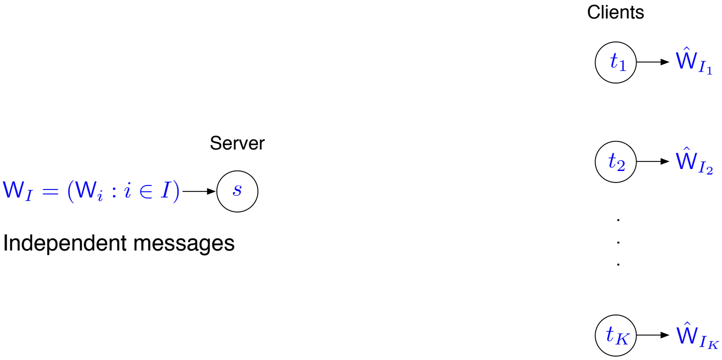

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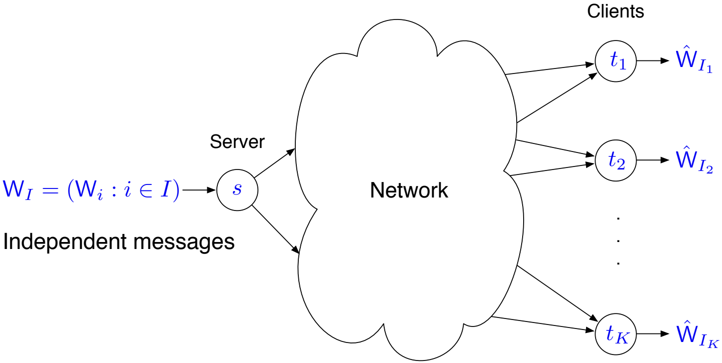
- Motivation: Content delivery over wired networks
- Network: A capacitated directed acyclic graph



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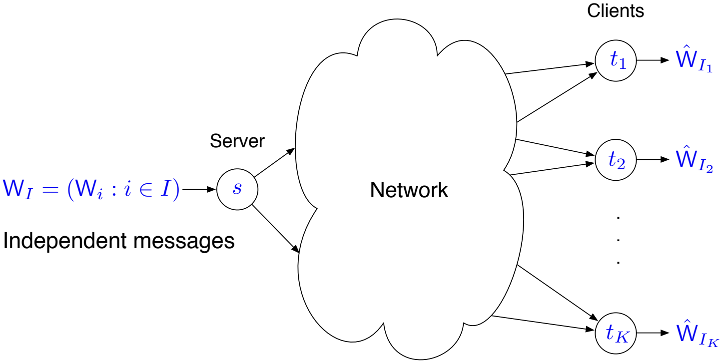
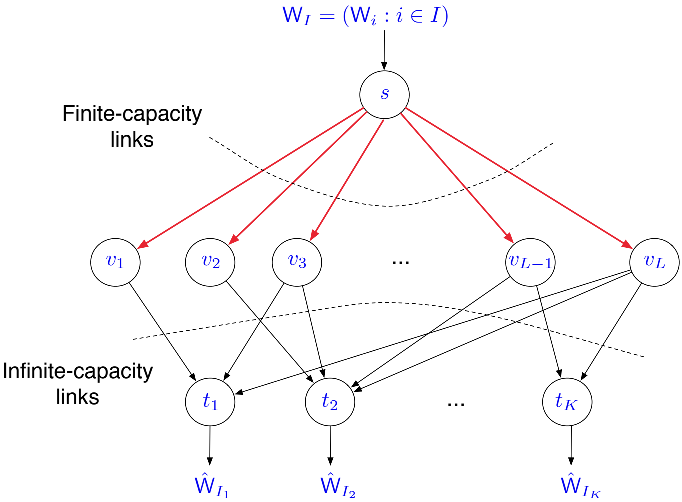


What is the capacity region $\{R_I = (R_i : i \in I)\}$?

Combination Networks

- Motivation: Distributed/cloud storage systems



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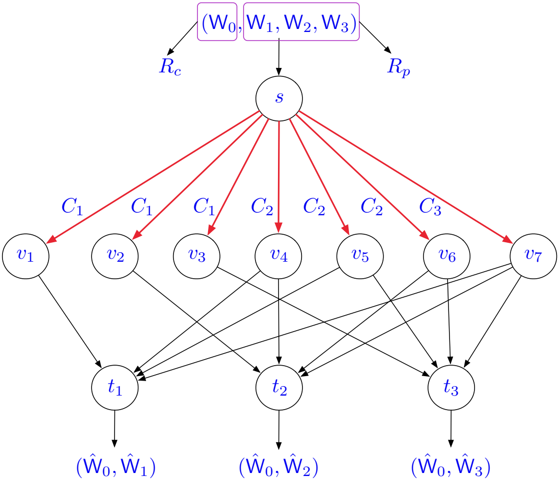
- Loose in general

Example 1: Symmetrical Combination Network with Three Sinks



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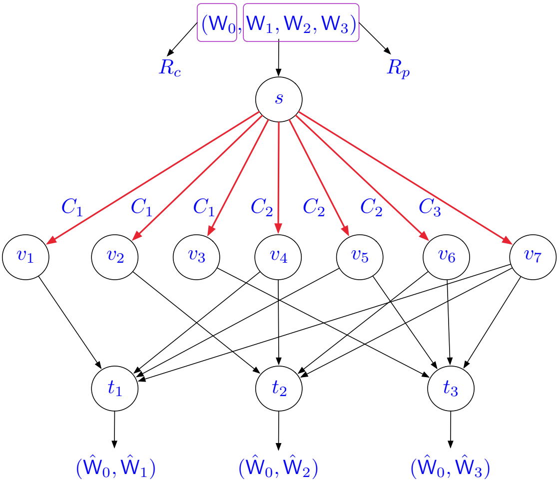

What is the capacity region $\{(R_c, R_p)\}$?

Capacity v.s. Cut-Set Outer Regions

- The cut-set outer region:

$$\begin{aligned} R_c + R_p &\leq C_1 + 2C_2 + C_3 \\ R_c + 2R_p &\leq 2C_1 + 3C_2 + C_3 \\ R_c + 3R_p &\leq 3C_1 + 3C_2 + C_3 \end{aligned}$$

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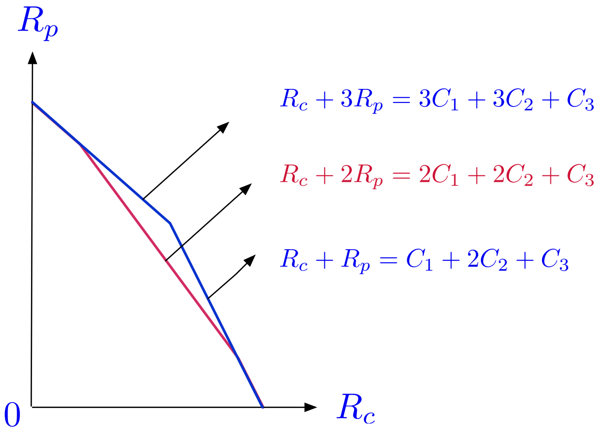

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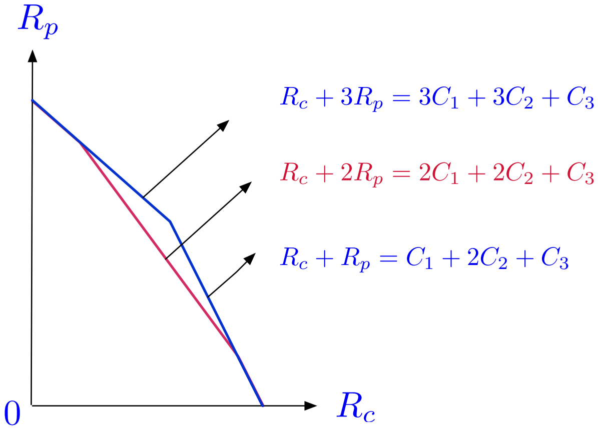

What is the nature of

$$R_c + 2R_p \leq 2C_1 + 2C_2 + C_3?$$

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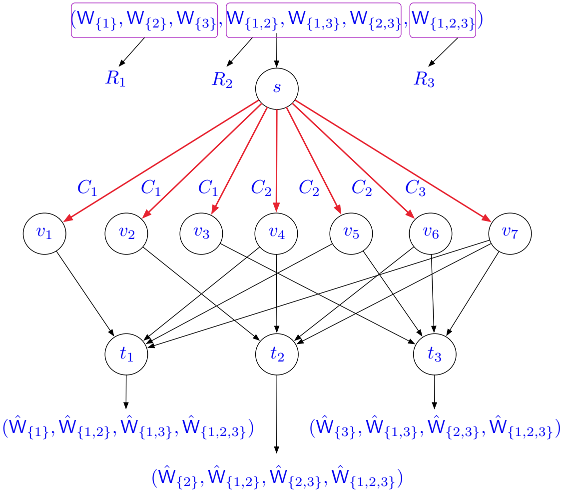

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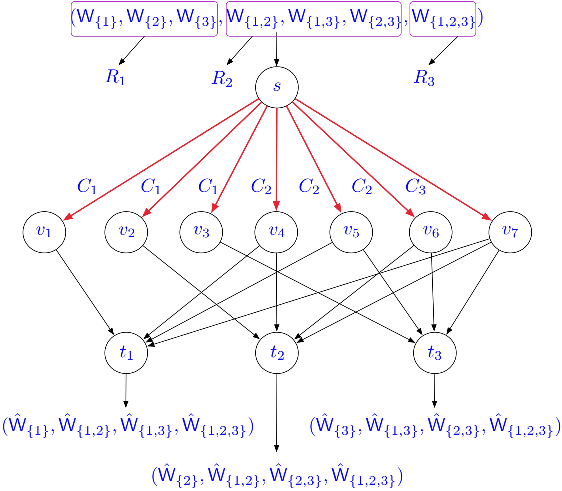

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- The capacity region (Tian 2011):

$$\begin{aligned} R_1 + 2R_2 + R_3 &\leq C_1 + 2C_2 + C_3 \\ 3R_1 + 3R_2 + R_3 &\leq 3C_1 + 3C_2 + C_3 \\ 2R_1 + 2R_2 + R_3 &\leq 2C_1 + 2C_2 + C_3 \\ 3R_1 + 6R_2 + 2R_3 &\leq 3C_1 + 6C_2 + 2C_3 \end{aligned}$$

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What is the nature of

$$2R_1 + 2R_2 + R_3 \leq 2C_1 + 2C_2 + C_3$$

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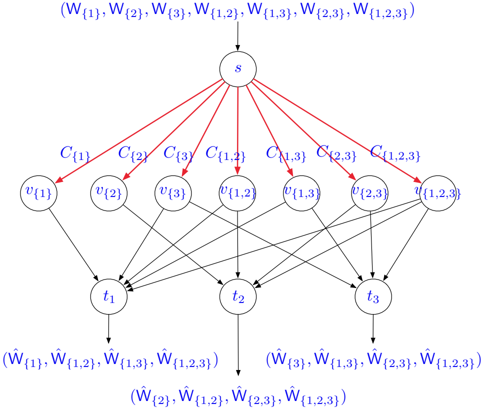
Example 3: General Combination Network with Three Sinks



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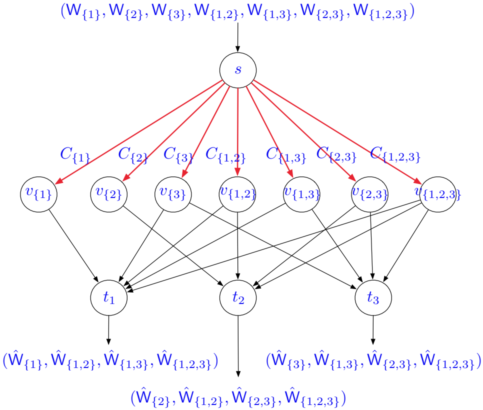


What is the capacity region

$$\left\{\left(R_{\{1\}}, R_{\{2\}}, R_{\{3\}}, R_{\{1,2\}}, R_{\{1,3\}}, R_{\{2,3\}}, R_{\{1,2,3\}}\right)\right\}?$$

The Capacity Region (Grokop-Tse 2008)

$$
\begin{aligned}
R_{\{1\}}+R_{\{1,2\}} + R_{\{1,3\}} + R_{\{1,2,3\}} &\leq C_{\{1\}} + C_{\{1,2\}} + C_{\{1,3\}} + C_{\{1,2,3\}} \\
R_{\{2\}}+R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,2,3\}} &\leq C_{\{2\}} + C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,2,3\}} \\
R_{\{3\}}+R_{\{1,3\}} + R_{\{2,3\}} + R_{\{1,2,3\}} &\leq C_{\{3\}} + C_{\{1,3\}} + C_{\{2,3\}} + C_{\{1,2,3\}} \\
R_{\{1\}} + R_{\{2\}}+R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,3\}} + R_{\{1,2,3\}} & \\
\leq C_{\{1\}} + C_{\{2\}} + C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,3\}} + C_{\{1,2,3\}} & \\
R_{\{2\}} + R_{\{3\}}+R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,3\}} + R_{\{1,2,3\}} & \\
\leq C_{\{2\}} + C_{\{3\}} + C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,3\}} + C_{\{1,2,3\}} & \\
R_{\{1\}} + R_{\{3\}}+R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,3\}} + R_{\{1,2,3\}} & \\
\leq C_{\{1\}} + C_{\{3\}} + C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,3\}} + C_{\{1,2,3\}} & \\
R_{\{1\}} + R_{\{2\}} + R_{\{3\}}+R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,3\}} + R_{\{1,2,3\}} & \\
\leq C_{\{1\}} + C_{\{2\}} + C_{\{3\}} + C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,3\}} + C_{\{1,2,3\}} &
\end{aligned}
$$

Seven cut-set bounds

The Capacity Region (Grokop-Tse 2008)

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+2R_{\{2,3\}}+R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+2C_{\{2,3\}}+C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+R_{\{2,3\}}+2R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+C_{\{2,3\}}+2C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+2R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq C_{\{1\}}+2C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

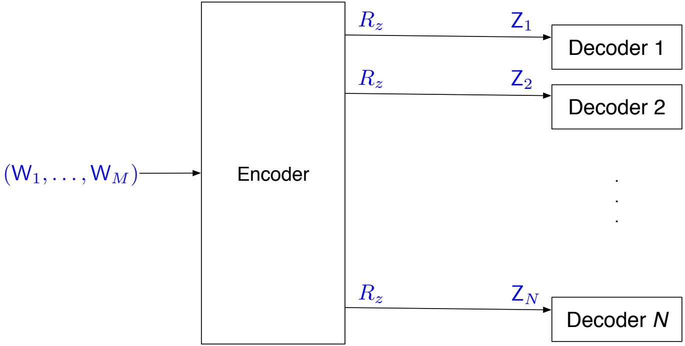
$$2R_{\{1\}}+R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

$$2R_{\{1\}}+2R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+2C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

$$2R_{\{1\}}+2R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+2C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

Eight non-cut-set bounds

Example 4: Caching for Cloud Storage (Maddah-Ali-Niesen 2012)



- Unit rates for all M messages

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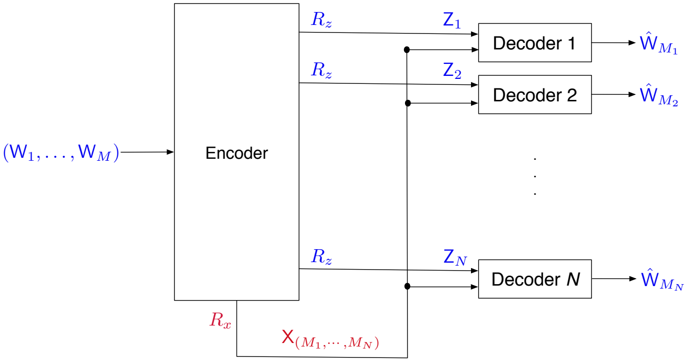

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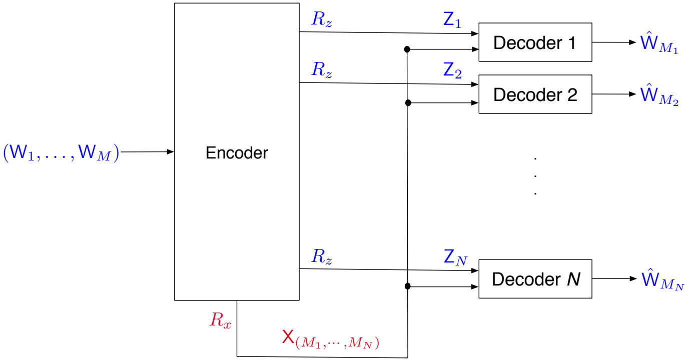

- Unit rates for all M messages
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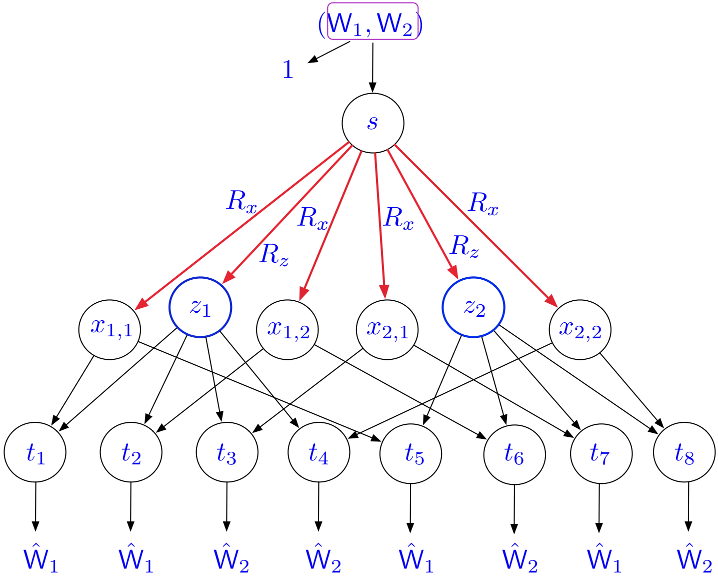
What is the admissible rate region $\{(R_x, R_z)\}$?

Equivalent Combination Network ($M = N = 2$)



Admissible Rate v.s. Cut-Set Inner Regions

- The cut-set inner region:

$$\begin{aligned} R_x + 2R_z &\geq 2 \\ 2R_x + R_z &\geq 2 \end{aligned}$$

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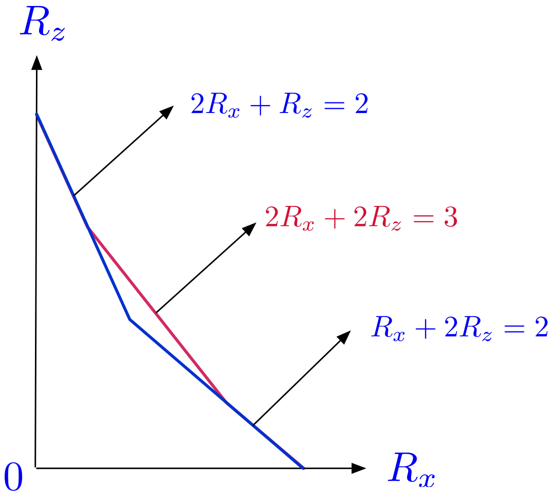
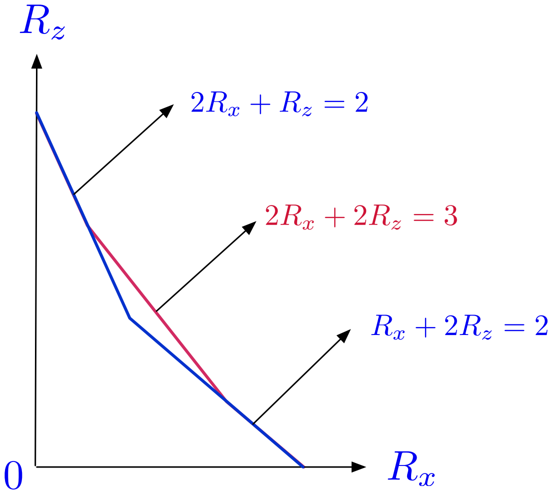
- The admissible rate region (Maddah-Ali-Niesen 2012):

$$\begin{aligned} R_x + 2R_z &\geq 2 \\ 2R_x + R_z &\geq 2 \\ 2R_x + 2R_z &\geq 3 \end{aligned}$$

Rate v.s. Cut-Set Inner Regions



Rate v.s. Cut-Set Inner Regions



What is the nature of $2R_x + 2R_z \geq 3$?

A Simple Observation

- Standard cut-set bound:

$$R(\cup_{k\in U} I_k) \leq C(A_U), \quad \forall U \subseteq [K]$$

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What about other set operations?

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

$$R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\ \leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))$$

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j) \\ \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

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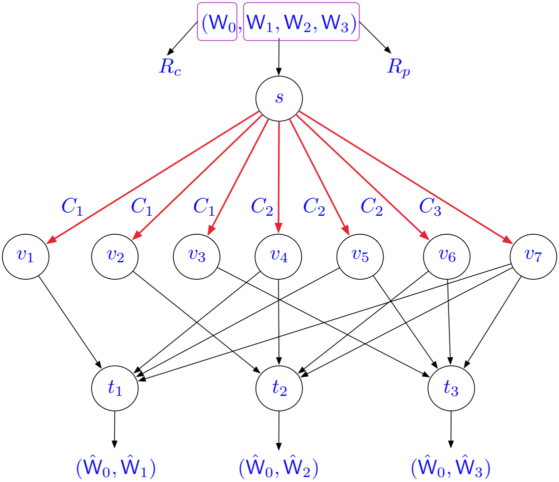

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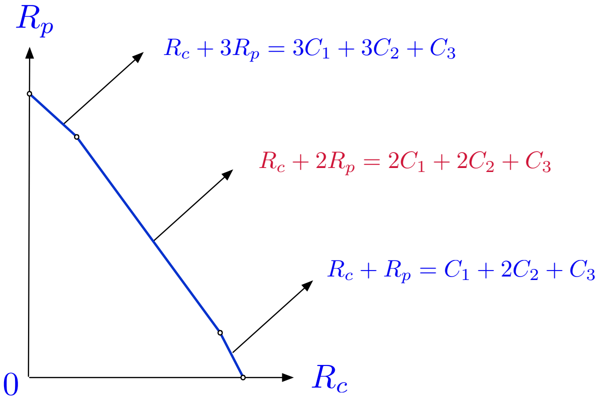
The Capacity Region

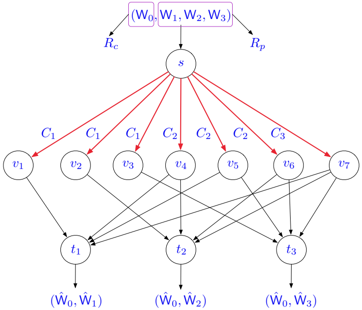


What is the nature of

$$R_c + 2R_p \leq 2C_1 + 2C_2 + C_3?$$

Example 1: Symmetrical Combination Network with Three Sinks

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$



$$R(I_1 \cup I_2 \cup I_3) = R_c + 3R_p, \quad R(I_1 \cap I_2 \cap I_3) = R_c$$

$$C(A_1 \cup A_2 \cup A_3) = 3C_1 + 3C_2 + C_3, \quad C(A_1 \cap A_2 \cap A_3) = C_3$$

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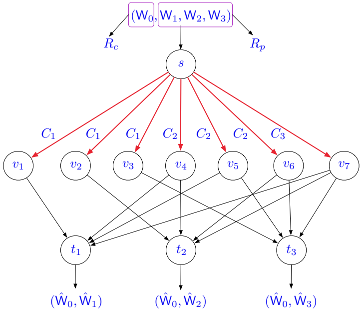
$$2(R_c + 3R_p) + R_c \leq 2(3C_1 + 3C_2 + C_3) + C_3$$



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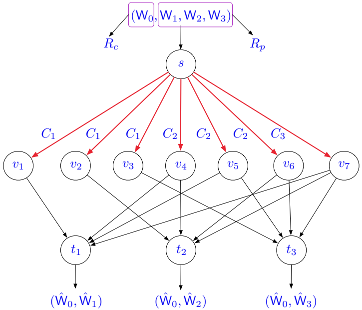
Example 1: Symmetrical Combination Network with Three Sinks

$$3R_c + 6R_p \leq 6C_1 + 6C_2 + 3C_3$$



$$R(I_1 \cup I_2 \cup I_3) = R_c + 3R_p, \quad R(I_1 \cap I_2 \cap I_3) = R_c$$
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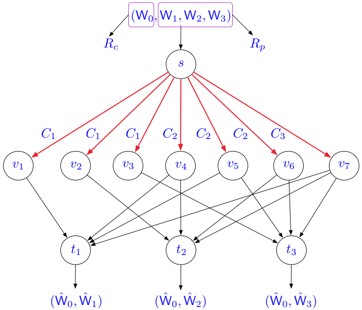
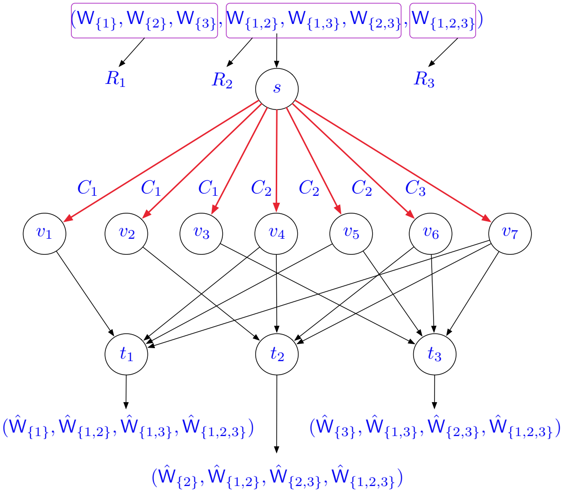
$$R_c + 2R_p \leq 2C_1 + 2C_2 + C_3$$



$$R(I_1 \cup I_2 \cup I_3) = R_c + 3R_p, \quad R(I_1 \cap I_2 \cap I_3) = R_c$$
$$C(A_1 \cup A_2 \cup A_3) = 3C_1 + 3C_2 + C_3, \quad C(A_1 \cap A_2 \cap A_3) = C_3$$

Example 2: Symmetrical Combination Network with Three Sinks



What is the capacity region $\{(R_1, R_2, R_3)\}$?

The Capacity Region (Tian 2011)

$$
\begin{aligned}
R_1 + 2R_2 + R_3 &\leq C_1 + 2C_2 + C_3 \\
3R_1 + 3R_2 + R_3 &\leq 3C_1 + 3C_2 + C_3 \\
2R_1 + 2R_2 + R_3 &\leq 2C_1 + 2C_2 + C_3 \\
3R_1 + 6R_2 + 2R_3 &\leq 3C_1 + 6C_2 + 2C_3
\end{aligned}
$$

What is the nature of

$$2R_1 + 2R_2 + R_3 \leq 2C_1 + 2C_2 + C_3$$

$$3R_1 + 6R_2 + 2R_3 \leq 3C_1 + 6C_2 + 2C_3?$$

Example 2: Symmetrical Combination Network with Three Sinks

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

Example 2: Symmetrical Combination Network with Three Sinks

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

$$2R_1 + 2R_2 + R_3 \leq 2C_1 + 2C_2 + C_3$$

Example 2: Symmetrical Combination Network with Three Sinks

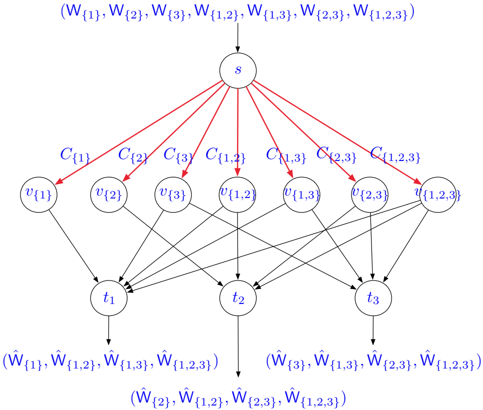
$$R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k))$$
$$\leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))$$

Example 2: Symmetrical Combination Network with Three Sinks

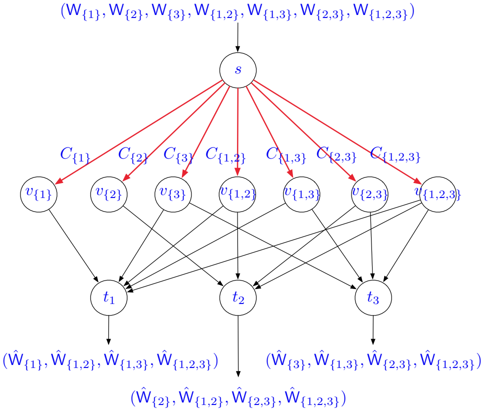
$$R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k))$$
$$\leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))$$

$$6R_1 + 3R_2 + 2R_3 \leq 6C_1 + 3C_2 + 2C_3$$

Example 3: General Combination Network with Three Sinks



Example 3: General Combination Network with Three Sinks



What is the capacity region

$$\{(R_{\{1\}}, R_{\{2\}}, R_{\{3\}}, R_{\{1,2\}}, R_{\{1,3\}}, R_{\{2,3\}}, R_{\{1,2,3\}})\}?$$

The Capacity Region (Grokop-Tse 2008)

$$R_{\{1\}}+R_{\{1,2\}}+R_{\{1,3\}}+R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{1,2\}}+C_{\{1,3\}}+C_{\{1,2,3\}}$$

$$R_{\{2\}}+R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,2,3\}} \leq C_{\{2\}}+C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,2,3\}}$$

$$R_{\{3\}}+R_{\{1,3\}}+R_{\{2,3\}}+R_{\{1,2,3\}} \leq C_{\{3\}}+C_{\{1,3\}}+C_{\{2,3\}}+C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+C_{\{1,2,3\}}$$

$$R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+R_{\{1,2,3\}} \leq C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{3\}}+R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{3\}}+C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+C_{\{1,2,3\}}$$

Seven cut-set bounds

The Capacity Region (Grokop-Tse 2008)

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+R_{\{2,3\}}+R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+C_{\{2,3\}}+C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+2R_{\{2,3\}}+R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+2C_{\{2,3\}}+C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+R_{\{1,2\}}+R_{\{2,3\}}+2R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+C_{\{1,2\}}+C_{\{2,3\}}+2C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+2R_{\{1,2,3\}} \leq C_{\{1\}}+C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+2C_{\{1,2,3\}}$$

$$R_{\{1\}}+2R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq C_{\{1\}}+2C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

$$2R_{\{1\}}+R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

$$2R_{\{1\}}+2R_{\{2\}}+R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+2C_{\{2\}}+C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

$$2R_{\{1\}}+2R_{\{2\}}+2R_{\{3\}}+2R_{\{1,2\}}+2R_{\{2,3\}}+2R_{\{1,3\}}+3R_{\{1,2,3\}} \leq 2C_{\{1\}}+2C_{\{2\}}+2C_{\{3\}}+2C_{\{1,2\}}+2C_{\{2,3\}}+2C_{\{1,3\}}+3C_{\{1,2,3\}}$$

Eight non-cut-set bounds

Example 3: General Combination Network with Three Sinks

$$
\begin{aligned}
&2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\
&\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)
\end{aligned}
$$

Example 3: General Combination Network with Three Sinks

$$
\begin{aligned}
&2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\
&\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)
\end{aligned}
$$

$$
\begin{aligned}
&2R_{\{1\}} + 2R_{\{2\}} + 2R_{\{3\}} + 2R_{\{1,2\}} + 2R_{\{2,3\}} + 2R_{\{1,3\}} + 3R_{\{1,2,3\}} \\
&\quad \leq 2C_{\{1\}} + 2C_{\{2\}} + 2C_{\{3\}} + 2C_{\{1,2\}} + 2C_{\{2,3\}} + 2C_{\{1,3\}} + 3C_{\{1,2,3\}}
\end{aligned}
$$

Example 3: General Combination Network with Three Sinks

$$
\begin{aligned}
&R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\
&\quad \leq C(A_i \cup A_i \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))
\end{aligned}
$$

Example 3: General Combination Network with Three Sinks

$$
\begin{aligned}
&R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\
&\quad \leq C(A_i \cup A_i \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))
\end{aligned}
$$

$$
\begin{aligned}
&R_{\{1\}} + R_{\{2\}} + R_{\{3\}} + 2R_{\{1,2\}} + 2R_{\{2,3\}} + 2R_{\{1,3\}} + 2R_{\{1,2,3\}} \\
&\quad \leq C_{\{1\}} + C_{\{2\}} + C_{\{3\}} + 2C_{\{1,2\}} + 2C_{\{2,3\}} + 2C_{\{1,3\}} + 2C_{\{1,2,3\}}
\end{aligned}
$$

Example 3: General Combination Network with Three Sinks

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

Example 3: General Combination Network with Three Sinks

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

$$R_{\{1\}} + R_{\{2\}} + R_{\{3\}} + 2R_{\{1,2\}} + R_{\{2,3\}} + R_{\{1,3\}} + 2R_{\{1,2,3\}}$$
$$\leq C_{\{1\}} + C_{\{2\}} + C_{\{3\}} + 2C_{\{1,2\}} + C_{\{2,3\}} + C_{\{1,3\}} + 2C_{\{1,2,3\}}$$
$$R_{\{1\}} + R_{\{2\}} + R_{\{3\}} + R_{\{1,2\}} + 2R_{\{2,3\}} + R_{\{1,3\}} + 2R_{\{1,2,3\}}$$
$$\leq C_{\{1\}} + C_{\{2\}} + C_{\{3\}} + C_{\{1,2\}} + 2C_{\{2,3\}} + C_{\{1,3\}} + 2C_{\{1,2,3\}}$$
$$R_{\{1\}} + R_{\{2\}} + R_{\{3\}} + R_{\{1,2\}} + R_{\{2,3\}} + 2R_{\{1,3\}} + 2R_{\{1,2,3\}}$$
$$\leq C_{\{1\}} + C_{\{2\}} + C_{\{3\}} + C_{\{1,2\}} + C_{\{2,3\}} + 2C_{\{1,3\}} + 2C_{\{1,2,3\}}$$

Example 3: General Combination Network with Three Sinks

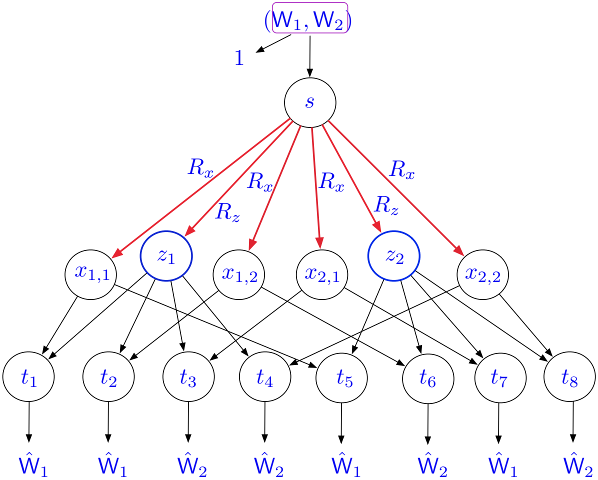
$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j)$$
$$\leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

Example 3: General Combination Network with Three Sinks

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j)$$
$$\leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

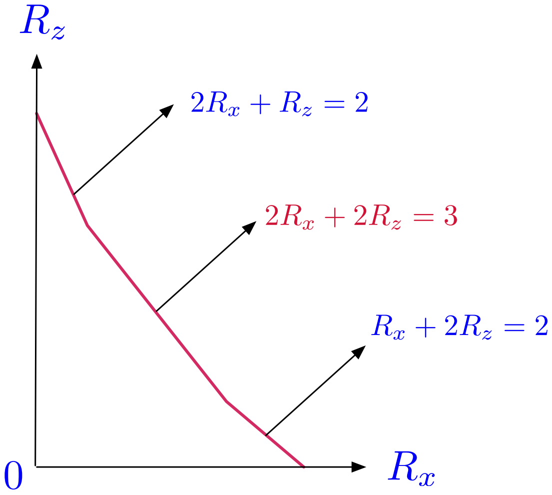
$$R_{\{1\}} + 2R_{\{2\}} + 2R_{\{3\}} + 2R_{\{1,2\}} + 2R_{\{2,3\}} + 2R_{\{1,3\}} + 3R_{\{1,2,3\}}$$
$$\leq C_{\{1\}} + 2C_{\{2\}} + 2C_{\{3\}} + 2C_{\{1,2\}} + 2C_{\{2,3\}} + 2C_{\{1,3\}} + 3C_{\{1,2,3\}}$$
$$2R_{\{1\}} + R_{\{2\}} + 2R_{\{3\}} + 2R_{\{1,2\}} + 2R_{\{2,3\}} + 2R_{\{1,3\}} + 3R_{\{1,2,3\}}$$
$$\leq 2C_{\{1\}} + C_{\{2\}} + 2C_{\{3\}} + 2C_{\{1,2\}} + 2C_{\{2,3\}} + 2C_{\{1,3\}} + 3C_{\{1,2,3\}}$$
$$2R_{\{1\}} + 2R_{\{2\}} + R_{\{3\}} + 2R_{\{1,2\}} + 2R_{\{2,3\}} + 2R_{\{1,3\}} + 3R_{\{1,2,3\}}$$
$$\leq 2C_{\{1\}} + 2C_{\{2\}} + C_{\{3\}} + 2C_{\{1,2\}} + 2C_{\{2,3\}} + 2C_{\{1,3\}} + 3C_{\{1,2,3\}}$$

Example 4: Caching for Cloud Storage ($M = N = 2$)



What is the admissible rate region $\{(R_x, R_z)\}$?

The Admissible Rate Region (Maddah-Ali-Niesen 2012)



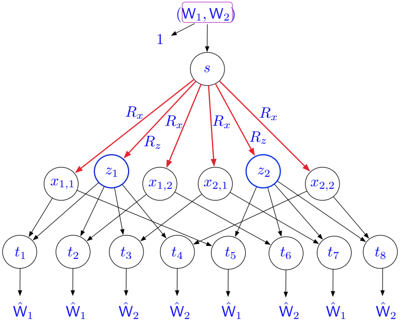
What is the nature of $2R_x + 2R_z \geq 3$?

Example 4: Caching for Cloud Storage ($M = N = 2$)

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

Example 4: Caching for Cloud Storage ($M = N = 2$)

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

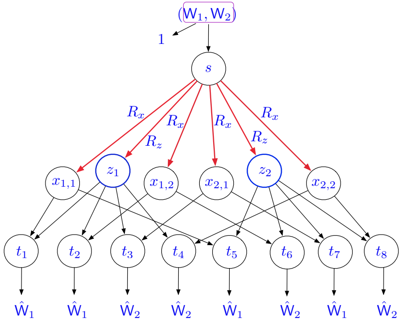


$$R(I_1 \cup I_3 \cup I_7) = 2, \quad R(I_1 \cap I_7) = 1$$

$$C(A_1 \cup A_3 \cup A_7) = 2R_x + 2R_z, \quad C(A_1 \cap A_7) = 0$$

Example 4: Caching for Cloud Storage ($M = N = 2$)

$$2 + 1 \leq (2R_x + 2R_z) + 0$$

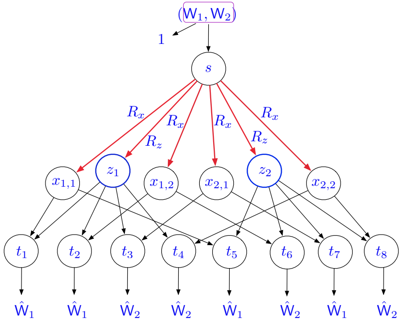


$$R(I_1 \cup I_3 \cup I_7) = 2, \quad R(I_1 \cap I_7) = 1$$

$$C(A_1 \cup A_3 \cup A_7) = 2R_x + 2R_z, \quad C(A_1 \cap A_7) = 0$$

Example 4: Caching for Cloud Storage ($M = N = 2$)

$$3 \le 2R_x + R_z$$



$$R(I_1 \cup I_3 \cup I_7) = 2, \quad R(I_1 \cap I_7) = 1$$

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Generalized Cut-Set Bounds Relating Three Basic Cuts

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

$$R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\ \leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))$$

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$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j) \\ \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

$$R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\ \leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))$$

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j) \\ \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

How to prove them?

From Graph to Information Theory

- Standard cut-set bounds:

$$R(\cup_{k \in U} I_k) \leq C(\cup_{k \in U} A_k), \quad \forall U \subseteq [K]$$

  – For any $k \in U$, W_{I_k} are functions of X_{A_k}
  – Of course, $W_{\cup_{k \in U} I_k}$ are functions of $X_{\cup_{k \in U} A_k}$

From Graph to Information Theory

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  - A graph-theoretic result

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  - A graph-theoretic result

- Generalized cut-set bounds:

$$\sum_i \alpha_i R(\Phi_i(I_1, \ldots, I_K)) \leq \sum_i \alpha_i C(\Phi_i(A_1, \ldots, A_K))$$

  - α_i: Positive reals
  - Φ_i: Set operators

From Graph to Information Theory

- Standard cut-set bounds:

$$R(\cup_{k\in U} I_k) \le C(\cup_{k\in U} A_k), \quad \forall U \subseteq [K]$$

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  - Of course, $W_{\cup_{k\in U} I_k}$ are functions of $X_{\cup_{k\in U} A_k}$
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- Generalized cut-set bounds:

$$\sum_i \alpha_i R(\Phi_i(I_1, \ldots, I_K)) \le \sum_i \alpha_i C(\Phi_i(A_1, \ldots, A_K))$$

  - α_i: Positive reals
  - Φ_i: Set operators
  - Need to exploit the relationship among different basic cuts via information theory

Shannon Entropy

- Entropy of a single random variable Z (Shannon 1948):

$$H(\mathsf{Z}) := \sum_{z \in \mathcal{Z}} p(z) \log \frac{1}{p(z)}$$

  - Measures the amount of information contained in Z
  - Always nonnegative

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  - View Z_S as a vector-valued random variable

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- Entropy of a collection of random variables $\mathsf{Z}_S = (\mathsf{Z}_i : i \in S)$:
  - View Z_S as a vector-valued random variable
  - Entropy as a set function

$$H_{\mathsf{Z}}(S') := H(\mathsf{Z}_{S'}), \quad \forall S' \subseteq S$$

  is monotonic and submodular

Submodular and Modular Functions

A function $f : 2^S \rightarrow \mathbb{R}$ is said to be a submodular function if

$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2), \quad \forall S_1, S_2 \subseteq S$$

is said to be a modular function if

$$f(S_1) + f(S_2) = f(S_1 \cup S_2) + f(S_1 \cap S_2), \quad \forall S_1, S_2 \subseteq S.$$

Diminishing Returns

A function $f : 2^S \to \mathbb{R}$ is said to be a submodular function if

$$f(A \cup C) - f(A) \leq f(B \cup C) - f(B)$$

for any subsets A, B and C of S such that $A \supseteq B$ and $C \cap A = \emptyset$.

Diminishing Returns

A function $f : 2^S \to \mathbb{R}$ is said to be a submodular function if

$$f(A \cup C) - f(A) \leq f(B \cup C) - f(B)$$

for any subsets A, B and C of S such that $A \supseteq B$ and $C \cap A = \emptyset$.

Submodular functions are “convex” set functions.

Entropy and Capacity Functions

- The entropy function

$$H_{\mathsf{Z}}(S') := H(\mathsf{Z}_{S'}), \quad S' \subseteq S$$

  is nonnegative, monotonic and submodular

- The capacity function

$$C(A') := \sum_{a \in A'} C_a, \quad A' \subseteq A$$

  is modular

Entropy and Capacity Functions

- The entropy function

$$H_{\mathsf{Z}}(S') := H(\mathsf{Z}_{S'}), \quad S' \subseteq S$$

  is nonnegative, monotonic and submodular

- The capacity function

$$C(A') := \sum_{a \in A'} C_a, \quad A' \subseteq A$$

  is modular

How to translate these facts into generalized cut-set bounds?

Extremal Subset Inequalities

A 5-tuple $(\{\alpha_i\}, \{\Phi_i\}, \{\beta_j\}, \{\Gamma_j^+\}, \{\Gamma_j^-\})$ where

- α_i, β_j: Positive reals
- Φ_i, Γ_j^+, Γ_j^-: Set operators

defines an extremal subset inequality if for any K subsets $S_1, \ldots, S_K$ of a ground set S and any set function $f : 2^S \to \mathbb{R}$

$$\sum_i \alpha_i f(\Phi_i(S_1, \ldots, S_K)) \leq \sum_j \beta_j \left(f(\Gamma_j^+(S_1, \ldots, S_K)) - f(\Gamma_j^-(S_1, \ldots, S_K)) \right)$$

whenever f is submodular, and

$$\sum_i \alpha_i f(\Phi_i(S_1, \ldots, S_K)) = \sum_j \beta_j \left(f(\Gamma_j^+(S_1, \ldots, S_K)) - f(\Gamma_j^-(S_1, \ldots, S_K)) \right)$$

whenever f is modular

A Meta Theorem

The generalized cut-set bound

$$\sum_i \alpha_i R(\Phi_i(I_1, \dots, I_K)) \leq \sum_i \alpha_i C(\Phi_i(A_1, \dots, A_K))$$

identified by $\{\alpha_i\}$ and $\{\Phi_i\}$ holds if there exist positive reals $\{\beta_j\}$ and set operators $\{\Gamma_j^+\}$ and $\{\Gamma_j^-\}$ such that:

- $(\{\alpha_i\}, \{\Phi_i\}, \{\beta_j\}, \{\Gamma_j^+\}, \{\Gamma_j^-\})$ defines an extremal subset inequality;
- $\{\Gamma_j^+\}$ are subset unions; and
- $\Gamma_j^+ \supseteq \Gamma_j^-$ for all j

Proof

$$\sum_i \alpha_i H_{\mathsf{Z}}(\Phi_i(S_1, \ldots, S_K)) \leq \sum_j \beta_j \left(H_{\mathsf{Z}}(\Gamma_j^+(S_1, \ldots, S_K)) - H_{\mathsf{Z}}(\Gamma_j^-(S_1, \ldots, S_K)) \right)$$

Proof

$$\sum_i \alpha_i H_{\mathsf{W},\mathsf{X}}(\Phi_i(I_1,\ldots,I_K),\Phi_i(A_1,\ldots,A_K))$$
$$\leq \sum_j \beta_j \Big(H_{\mathsf{W},\mathsf{X}}(\Gamma_j^+(I_1,\ldots,I_K),\Gamma_j^+(A_1,\ldots,A_K)) -$$
$$H_{\mathsf{W},\mathsf{X}}(\Gamma_j^-(I_1,\ldots,I_K),\Gamma_j^-(A_1,\ldots,A_K)) \Big)$$

Proof

$$\sum_i \alpha_i H_{\mathsf{W}}(\Phi_i(I_1, \ldots, I_K))$$

$$\leq \sum_j \beta_j \left(H_{\mathsf{W},\mathsf{X}}(\Gamma_j^+(I_1, \ldots, I_K), \Gamma_j^+(A_1, \ldots, A_K)) - \right.$$

$$\left. H_{\mathsf{X}}(\Gamma_j^-(A_1, \ldots, A_K)) \right)$$

Proof

$$\sum_i \alpha_i H_{\mathsf{W}}(\Phi_i(I_1, \ldots, I_K)) \\ \leq \sum_j \beta_j \left(H_{\mathsf{X}}(\Gamma_j^+(A_1, \ldots, A_K)) - H_{\mathsf{X}}(\Gamma_j^-(A_1, \ldots, A_K)) \right)$$

Proof

$$\sum_i \alpha_i H_{\mathsf{W}}(\Phi_i(I_1, \ldots, I_K)) \leq \sum_j \beta_j H_{\mathsf{X}}(\Gamma_j^+(A_1, \ldots, A_K) \setminus \Gamma_j^-(A_1, \ldots, A_K))$$

Proof

$$\sum_i \alpha_i R(\Phi_i(I_1, \ldots, I_K))$$

$$\leq \sum_j \beta_j C(\Gamma_j^+(A_1, \ldots, A_K) \setminus \Gamma_j^-(A_1, \ldots, A_K))$$

Proof

$$\sum_i \alpha_i R(\Phi_i(I_1, \ldots, I_K)) \\ \leq \sum_j \beta_j \left(C(\Gamma_j^+(A_1, \ldots, A_K)) - C(\Gamma_j^-(A_1, \ldots, A_K)) \right)$$

Proof

$$\sum_i \alpha_i R(\Phi_i(I_1, \ldots, I_K)) \leq \sum_i \alpha_i C(\Phi_i(A_1, \ldots, A_K))$$

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$
\begin{aligned}
&2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\
&\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)
\end{aligned}
$$

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

$$\begin{aligned} &2f(S_i \cup S_j \cup S_k) + f(S_i \cap S_j \cap S_k) \\ &\quad \leq f(S_i) + f(S_j) + f(S_k) + (f(S_i) - f(S_i \cap (S_j \cup S_k))) + \\ &\qquad (f(S_j) - f(S_j \cap (S_i \cup S_k)) + (f(S_k) - f(S_k \cap (S_i \cup S_j)) \end{aligned}$$

whenever f is submodular, and

$$\begin{aligned} &2f(S_i \cup S_j \cup S_k) + f(S_i \cap S_j \cap S_k) \\ &\quad = f(S_i) + f(S_j) + f(S_k) + (f(S_i) - f(S_i \cap (S_j \cup S_k))) + \\ &\qquad (f(S_j) - f(S_j \cap (S_i \cup S_k)) + (f(S_k) - f(S_k \cap (S_i \cup S_j)) \end{aligned}$$

whenever f is modular

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$
\begin{aligned}
&R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\
&\quad \leq C(A_i \cup A_i \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k))
\end{aligned}
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Generalized Cut-Set Bounds Relating Three Basic Cuts

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\end{aligned}
$$

$$
\begin{aligned}
&f(S_i \cup S_j \cup S_k) + f((S_i \cap S_j) \cup (S_i \cap S_k) \cup (S_j \cap S_k)) \\
&\quad \leq (f(S_i) - f(S_i \cap S_j \cap S_k)) + f(S_j) + f(S_k)
\end{aligned}
$$

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$$
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Generalized Cut-Set Bounds Relating Three Basic Cuts

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Generalized Cut-Set Bounds Relating Three Basic Cuts

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j) \leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j)$$

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whenever f is submodular, and

$$\begin{aligned} f(S_i \cup S_j \cup S_k) + f(S_i \cap S_j) \\ = f(S_i) + f(S_j) + (f(S_k) - f(S_k \cap (S_i \cup S_j)) \end{aligned}$$

whenever f is modular

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j)$$
$$\leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

Generalized Cut-Set Bounds Relating Three Basic Cuts

$$R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) + R(I_i \cup I_j)$$
$$\leq C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) + C(A_i \cup A_j)$$

$$f(S_i \cup S_j \cup S_k) + f(S_i \cap S_j \cap S_k) + f(S_i \cup S_j)$$
$$\leq f(S_i) + f(S_j) + f(S_k) + (f(S_i) - f(S_i \cap (S_j \cup S_k))) +$$
$$(f(S_j) - f(S_j \cap (S_i \cup S_k))$$

whenever f is submodular, and

$$f(S_i \cup S_j \cup S_k) + f(S_i \cap S_j \cap S_k) + f(S_i \cup S_j)$$
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$$(f(S_j) - f(S_j \cap (S_i \cup S_k))$$

whenever f is modular

From Two-Way to Multiway Submodularity

- Two-way submodularity:

$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2), \quad \forall S_1, S_2 \subseteq S$$

From Two-Way to Multiway Submodularity

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$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2), \quad \forall S_1, S_2 \subseteq S$$

- Multiway submodularity (Harvey-Kleinberg-Lehman 2006):

$$\sum_{k \in U} f(S_k) \geq \sum_{r=1}^{|U|} f(S^{(r)}(U))$$

  - $U \subseteq [K]$: An index set
  - $S^{(r)}(U) := \cup_{\{U' \subseteq U : |U'| = r\}} \cap_{k \in U'} S_k$
  - $\cup_{k \in U} S_k =: S^{(1)}(U) \supseteq S^{(2)}(U) \supseteq \cdots \supseteq S^{(|U|)}(U) := \cap_{k \in U} S_k$

Generalized Cut-Set Bounds Relating K Basic Cuts

$$
\begin{aligned}
&2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\
&\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)
\end{aligned}
$$

Generalized Cut-Set Bounds Relating K Basic Cuts

$$\begin{aligned} 2R(I^{(1)}(\{i,j,k\})) + R(I^{(3)}(\{i,j,k\})) \\ \leq 2C(A^{(1)}(\{i,j,k\})) + C(A^{(3)}(\{i,j,k\})) \end{aligned}$$

Generalized Cut-Set Bounds Relating K Basic Cuts

$$2R(I^{(1)}(\{i,j,k\})) + R(I^{(3)}(\{i,j,k\})) \leq 2C(A^{(1)}(\{i,j,k\})) + C(A^{(3)}(\{i,j,k\}))$$

For any $m = 1, \ldots, |U|$,

$$mR(I^{(1)}(U)) + \sum_{r=m+1}^{|U|} R(I^{(r)}(U)) \leq mC(A^{(1)}(U)) + \sum_{r=m+1}^{|U|} C(A^{(r)}(U))$$

Proof

Assume that f is a submodular function:

$$\sum_{r=1}^{|U|} f(S^{(r)}(U)) \leq \sum_{k \in U} f(S_k)$$

Proof

Assume that f is a submodular function:

$$f(S^{(1)}(U)) + \sum_{r=m+1}^{|U|} f(S^{(r)}(U)) \leq \sum_{k \in U} f(S_k) - \sum_{r=2}^{m} f(S^{(r)}(U))$$

Proof

Assume that f is a submodular function:

$$f(S^{(1)}(U)) + \sum_{r=m+1}^{|U|} f(S^{(r)}(U)) \leq \sum_{k\in U} f(S_k) - \sum_{r=2}^{m} f(S^{(r)}(U))$$

Adding $(m-1)f(S^{(1)}(U))$ on both sides:

$$mf(S^{(1)}(U)) + \sum_{r=m+1}^{|U|} f(S^{(r)}(U)) \leq \sum_{k\in U} f(S_k) + \sum_{r=2}^{m} \left(f(S^{(1)}(U)) - f(S^{(r)}(U)) \right)$$

Proof

Assume that f is a submodular function:

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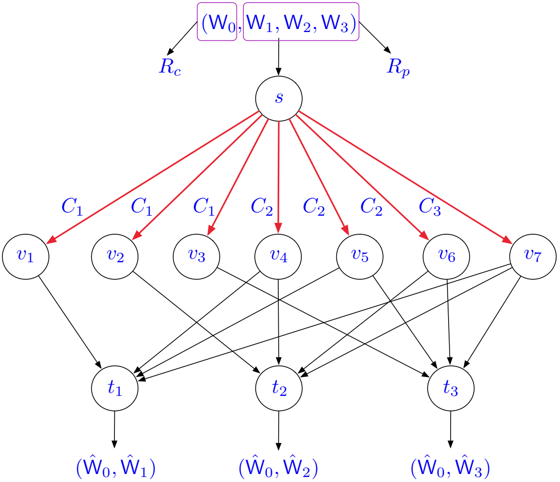
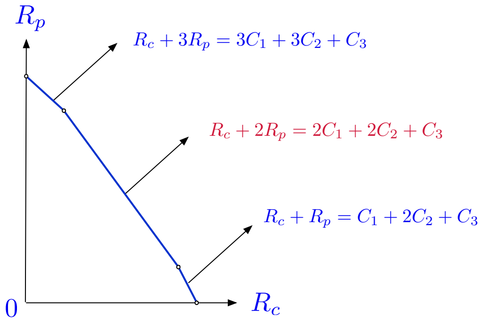
Equalities hold when f is a modular function

Example 1: Symmetrical Combination Network with Three Sinks



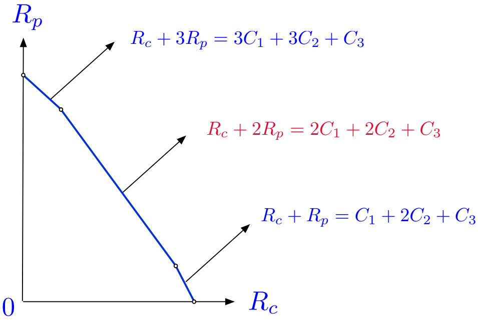
What is the capacity region $\{(R_c, R_p)\}$?

The Capacity Region



$$2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k)$$

The Capacity Region



$$\begin{aligned} &2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ &\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) \end{aligned}$$

What about K sinks?

Generalized Cut-Set Bounds

- $U = [K]$:

$$mR(I^{(1)}([K])) + \sum_{r=m+1}^{|U|} R(I^{(r)}([K])) \leq mC(A^{(1)}([K])) + \sum_{r=m+1}^{|U|} C(A^{(r)}([K]))$$

for any $m = 1, \ldots, K$

Generalized Cut-Set Bounds

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$$mR(I^{(1)}([K])) + \sum_{r=m+1}^{|U|} R(I^{(r)}([K])) \leq mC(A^{(1)}([K])) + \sum_{r=m+1}^{|U|} C(A^{(r)}([K]))$$

  for any $m = 1, \ldots, K$

- With K sinks,

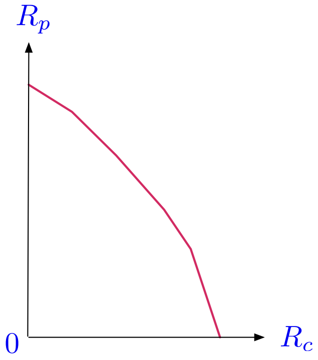
$$R(I^{(r)}([K])) = \begin{cases} R_c + KR_p, & r = 1 \\ R_c, & r = 2, \ldots, K \end{cases}$$

$$C(A^{(r)}([K])) = \sum_{j=r}^{K} \binom{K}{j} C_j$$

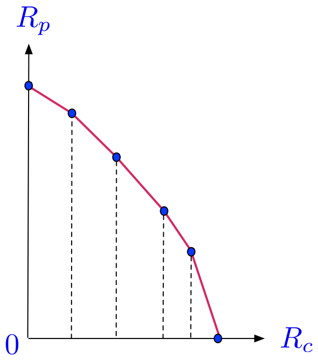
Generalized Cut-Set Bounds

For any $m = 1, \ldots, K$,

$$K(R_c + mR_p) \le m \sum_{j=1}^{m} \binom{K}{j} C_j + \sum_{j=m+1}^{K} j \binom{K}{j} C_j$$



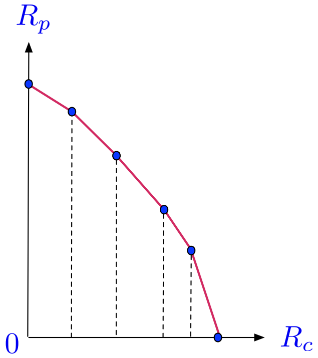
The Capacity Region



- A total of $K+1$ corner points:

$$\left(\sum_{j=r}^{K}\binom{K-1}{j-1}C_j, \frac{1}{K}\sum_{j=1}^{r-1}\binom{K}{j}C_j\right), \quad r = 1, \ldots, K+1$$

The Capacity Region



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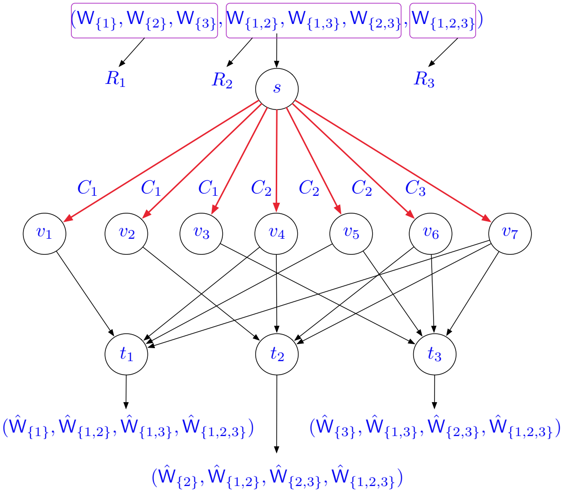
$$\left(\sum_{j=r}^{K} \binom{K-1}{j-1} C_j, \frac{1}{K} \sum_{j=1}^{r-1} \binom{K}{j} C_j \right), \quad r = 1, \ldots, K+1$$

- All corner points achievable by maximum-distance separable (MDS) codes!

Example 2: Symmetrical Combination Network with Three Sinks



What is the capacity region $\{(R_1, R_2, R_3)\}$?

The Capacity Region (Tian 2011)

$$\begin{aligned} R_1 + 2R_2 + R_3 &\leq C_1 + 2C_2 + C_3 \\ 3R_1 + 3R_2 + R_3 &\leq 3C_1 + 3C_2 + C_3 \\ 2R_1 + 2R_2 + R_3 &\leq 2C_1 + 2C_2 + C_3 \\ 3R_1 + 6R_2 + 2R_3 &\leq 3C_1 + 6C_2 + 2C_3 \end{aligned}$$

$$\begin{aligned} &2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\ &\quad \leq 2C(A_i \cup A_j \cup A_k) + C(A_i \cap A_j \cap A_k) \end{aligned}$$

$$\begin{aligned} &R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\ &\quad \leq C(A_i \cup A_j \cup A_k) + C((A_i \cap A_j) \cup (A_i \cap A_k) \cup (A_j \cap A_k)) \end{aligned}$$

The Capacity Region (Tian 2011)

$$
\begin{aligned}
R_1 + 2R_2 + R_3 &\leq C_1 + 2C_2 + C_3 \\
3R_1 + 3R_2 + R_3 &\leq 3C_1 + 3C_2 + C_3 \\
2R_1 + 2R_2 + R_3 &\leq 2C_1 + 2C_2 + C_3 \\
3R_1 + 6R_2 + 2R_3 &\leq 3C_1 + 6C_2 + 2C_3
\end{aligned}
$$

$$
\begin{aligned}
&2R(I_i \cup I_j \cup I_k) + R(I_i \cap I_j \cap I_k) \\
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$$

$$
\begin{aligned}
&R(I_i \cup I_j \cup I_k) + R((I_i \cap I_j) \cup (I_i \cap I_k) \cup (I_j \cap I_k)) \\
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\end{aligned}
$$

What about K sinks?

Generalized Cut-Set Bounds

Let U be a subset of $[K]$, and let Q be a subset of $\{2, \ldots, |U|\}$. We have

$$\sum_{r=1}^{|U|} \beta_Q(r) R(I^{(r)}(U)) \leq \sum_{r=1}^{|U|} \beta_Q(r) C(A^{(r)}(U))$$

where

$$\beta_Q(r) = \begin{cases} 0, & \text{if } r \in Q \\ \prod_{\{q \in Q : q < r\}} (q-1) \prod_{\{q \in Q : q > r\}} q, & \text{if } r \notin Q \end{cases}$$

Generalized Cut-Set Bounds

- $U = [K]$:

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- With K sinks,

$$R(I^{(r)}([K])) = \sum_{j=r}^{K} \binom{K}{j} R_j$$

$$C(A^{(r)}([K])) = \sum_{j=r}^{K} \binom{K}{j} C_j$$

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- A polyhedral cone constrained by 2^{K-1} half-planes

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- A polyhedral cone constrained by 2^{K-1} half-planes
- Conjecture (numerically verified for small K): Generated by $2(K-1)$ "achievable" extremal rays

Generalized Cut-Set Bounds Relating K Basic Cuts

Let G, U and T be subsets of $[K]$ such that $G \supseteq U$, and let Q be a subset of $\{2, \ldots, |U|\}$. Let $(r_q : q \in Q)$ be a sequence of integers from $[|T|]$ such that $S^{(q)}(U) \subseteq S^{(r_q)}(T)$ for any K subsets $S_1, \ldots, S_k$ of S and any $q \in Q$. We have

$$R(I^{(1)}(G)) + \sum_{r \in \{2,\ldots,|U|\} \setminus Q} R(I^{(r)}(U)) + \sum_{q \in Q} \sum_{r=1}^{r_q} \alpha_Q(q,r) R(I^{(r)}(T))$$

$$\leq C(A^{(1)}(G)) + \sum_{r \in \{2,\ldots,|U|\} \setminus Q} C(A^{(r)}(U)) + \sum_{q \in Q} \sum_{r=1}^{r_q} \alpha_Q(q,r) C(A^{(r)}(T))$$

where

$$\alpha_Q(q,r) = \begin{cases} 0, & \text{if } r \in Q \\ \dfrac{\prod_{\{p \in Q: p<r\}}(p-1) \prod_{\{p \in Q: r<p \leq r_q\}} p}{r_q \prod_{\{p \in Q: p \leq r_q\}}(p-1)}, & \text{if } r \notin Q \end{cases}$$

Summary

- Generalized cut-set bounds for broadcast networks
  - To capture the structure of the network via set operators

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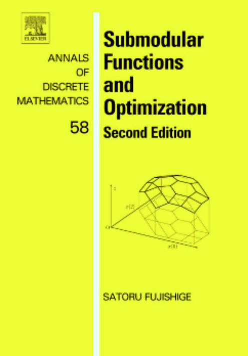
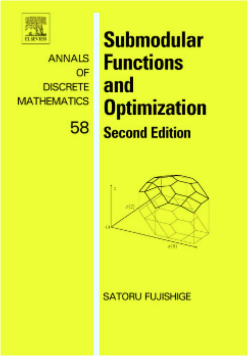
  - To capture the structure of the network via set operators
- A mechanism for depositing submodular function optimization results into network coding bounds



Summary

- Generalized cut-set bounds for broadcast networks
  - To capture the structure of the network via set operators
- A mechanism for depositing submodular function optimization results into network coding bounds



- Much more work to be done!